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PLANE LINEAR PROBLEM OF THE IMMERSION OF AN ELASTIC PLATE IN AN IDEAL INCOMPRESSIBLE FLUID

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A plane unsteady-state linear problem of the immersion of an elastic plate of finite length in an ideal incompressible weightless fluid is considered. The deflection of the plate and the velocity of its points are known at the initial moment of time. The fluid occupies the lower half-plane, and its boundary outside the plate is free. The plate which is the bottom of a structure immersed in the fluid with a constant velocity is modeled by an Euler beam. At the initial stage of immersion, when the displacement of the structure is much smaller than the length of the plate, the plate deflection and the distribution of bending stresses in it are determined. The model used allows one to estimate the maximum stresses occurring in the elastic plate during its impact on water and to predict the moment and site of its occurrence. Calculations are performed under the conditions of the experiment carried out in MARINTEX (Norway). Qualitative agreement between the numerical and experimental results is shown.

Introduction. A plane unsteady-state problem of the immersion of an elastic plate of finite length in an ideal incompressible weightless fluid is considered. At the initial moment ($t' = 0$), the fluid occupies the lower half-plane ($y' \leqslant 0$); the segments of its boundary $-L < x' < L$ and $y' = 0$ correspond to the elastic plate, and the segments $x' > L$, $x' < -L$, and $y' = 0$ to the free boundary of the fluid (Fig. 1). The dimensional variables are primed. The plate is hinged to a structure which is being immersed in the fluid with constant velocity V. The impact phenomena, which are connected with the beginning of the motion, determine the initial plate deflection $w'(x', 0)$ and the velocity of its points $(\partial w'/\partial t)(x', 0)$, which are assumed to be known and are denoted by $w_0'(x')$ and $w_1'(x')$, respectively. At the initial stage of immersion, when the structure is displaced to a much smaller extent than the length of the plate, one needs to determine the plate deflection and the distribution of bending stresses in it.

The problem is considered within the framework of a linear approximation. The fluid flow is assumed to be plane and potential. The plate is modeled by an Euler beam, and the bending stresses in the transverse direction are assumed to be negligible.

The impact by a shallow wave on a plate of finite size is divided into two stages [1]. At the first (impact) stage, the plate is wetted only partially, and the hydrodynamic loads on the plate are great and depend on the rate of expansion of the region of contact between the plate and the fluid. Generally, for shallow waves, this stage is short, and the stresses in the plate do not reach maximum values. At the second stage (immersion), the plate is wetted completely and continues to be immersed in the fluid. Here, the hydrodynamic loads on the plate are already insignificant and cannot be classified as impact. The plate vibrates mainly owing to the potential energy of elastic strains and to the kinetic energy accumulated in the plate during the impact stage. At both stages, the fluid boundary can be replaced by a plane boundary if the wave is quite shallow, and the depth of immersion of the plate is small compared with its size. The last remark explains the problem formulation for the second stage considered in the present work as a problem of immersion of a floating plate in an ideal weightless fluid for which the initial deflection and the velocity distribution are given.

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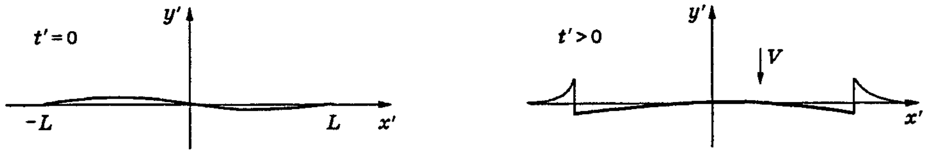

Fig. 1

The investigations of the interaction between elastic structures and the fluid were reviewed by Grigolyuk and Gorshkov [2]. Lotov [3] solved the problem of the impact of an elastic plate on the fluid surface within the framework of Sedov's impact theory. This theory does not take into account the specifics of the fluid flow at the impact stage and assumes that all points of the plate come in contact with the fluid simultaneously. This assumption is valid for infinitely long waves. It is shown that immediately after the moment of impact, the plate deflection can be ignored, and the vertical velocities of the points of the plate do not depend on its rigidity and the support conditions for its ends.

In the present work, another approach that is applicable to the case of waves of moderate length is used. The plate deflection and the vertical velocity of its points at the impact stage are first determined within the framework of the Wagner theory [4] without additional simplifying assumptions. The velocity of the plate points and its deflection at the end of the impact stage are used as the initial data for numerical calculation of the elastic behavior of the plate at the stage of immersion.

In a linear approximation, the strain of the fluid boundary is not taken into account, and the boundary conditions are used on the line $y' = 0$ and are linearized [5]. The fluid flow is uniquely described by the velocity potential $\varphi'(x', y', t')$, which satisfies the Laplace equation in the lower half-plane ($y' < 0$) and is zero on the segments of the free boundary $|x'| > L$ and $y' = 0$. In addition, the vertical velocity of the liquid particles on the elastic plate $(\partial\varphi'/\partial y')(x', 0, t')$ coincides with the absolute velocity of the points of the plate $-V + (\partial w'/\partial t')(x', t')$ ($|x'| < L$ and $t' > 0$), whereas the fluid pressure $p'(x', y', t')$ is connected with the velocity potential by the relation $p' = -\rho(\partial\varphi'/\partial t')$, where ρ is the density of the fluid.

It is assumed that the elastic strains of the plate are described by the Euler equation

$$M_B \frac{\partial^2 w'}{\partial t'^2} + EJ \frac{\partial^4 w'}{\partial x'^4} = p'(x', 0, t') \tag{1}$$

and are determined by the initial data

$$w'(x', 0) = w'_0(x'), \qquad \frac{\partial w'}{\partial t'}(x', 0) = w'_1(x'), \tag{2}$$

by the boundary conditions

$$w'(\pm L, 0) = 0, \qquad \frac{\partial^2 w'}{\partial x'^2}(\pm L, 0) = 0, \tag{3}$$

and the hydrodynamic loads on the plate. Here M_B is the mass of the beam of unit length, J is the moment of inertia of its cross section, and E is Young's modulus. The Euler model includes only bending stresses along the plate. The stresses in the transverse and longitudinal directions are not taken into account.

The problem consists of two mutually connected parts: a hydrodynamic part (the determination of the fluid flow and the pressure distributions over the known velocities of immersion of the points of the plate) and an elastic part (the determination of the strains in the plate with the use of the known initial data and distribution of the hydrodynamic pressure along the plate). It is convenient to write the hydrodynamic part of the linearized problem relative to the pressure $p'(x', y', t')$. The function $p'(x', y', t')$ is harmonic in the lower half-plane $y' < 0$ and is zero at the free boundary by virtue of the dynamic condition. On the segment $|x'| < L$ and $y' = 0$, after the no-flow condition has been differentiated with respect to time, it gives $(\partial p'/\partial y')(x', 0, t') = -\rho(\partial^2 w'/\partial t'^2)(x', t')$ for $t' > 0$. One can see that, at the stage of immersion, which follows the impact stage, the pressure profile depends only on the acceleration of the points of the plate, rather than the velocity of immersion if the latter remains constant.

Below, dimensionless variables are used. Analysis of the behavior of the elastic plate at the impact stage [6] shows that $w_0'(x') = (L^2/R)w_0(x'/L)$, where R is the radius of curvature in the crest of the wave, and $w_1'(x') = Vw_1(x'/L)$. The functions $w_0(x)$ and $w_1(x)$ are limited for $|x| < 1$, where $x = x'/L$. The quantity $T = [\rho L^5/(EJ)]^{1/2}$, which has the order of the period of oscillations of the first mode for a plate floating on the surface of a weightless fluid [1], is used as the time scale, L as the scale of length, and the product VT as the scale of strains at the fluid–plate boundary. The other scales are the derivatives of those indicated above: $\rho VL/T$ is used for the pressure, and VL for the velocity potential.

The linear approximation is formally true in the case where the depth of immersion of the plate is much smaller than its length: $VT \ll L$. With allowance for the definition of the time scale T, it is convenient to write the last condition in the form

$$V \ll \sqrt{EJ/(\rho L^3)}, \tag{4}$$

where the right-hand side has the dimension of velocity and is denoted by V_p. For example, for the experiments [1] carried out in MARINTEX (Norway) with a $1 \times 0.5 \times 0.008$-m steel plate, we have $E = 21 \cdot 10^{10}$ N/m^2, $h = 8$ mm, $J = h^3/12 = 4.266 \cdot 10^{-8}$ m^3, $\rho = 1000$ kg/m^3, $L = 25$ cm, and $V_p \approx 24$ m/sec. In this case, the inequality (4) means that the height at which the plate falls on the water surface should be much smaller than 30 m. Only in this case can the correspondence be reached between the calculations carried out within the framework of a linear approximation and the experimental data. In the experiments, the height of falling of the plate was 50 cm. Here, the use of the linear approximation is justified for the following scales of the immersion characteristics: the time is 0.01 sec, the length is 25 cm, the velocity is 2.5 m/sec, the pressure is $7.825 \cdot 10^4$ N/m^2, and the displacements are 3.13 mm. Using the method of [6], for the central impact by a wave with a radius of curvature in the crest of $R = 10.2$ m, the functions $w_0(x)$ and $w_1(x)$, which specify the initial conditions for the stage of immersion are determined, and the duration of the impact stage was found to be $8.5 \cdot 10^{-4}$ sec.

Formulation of the Problem. In dimensionless variables, relative to the pressure in the fluid $p(x, y, t)$ and the plate deflection $w(x, t)$, the problem has the form

$$p_{xx} + p_{yy} = 0 \quad (y < 0); \tag{5}$$

$$p = 0 \qquad (y = 0, \quad |x| > 1); \tag{6}$$

$$\frac{\partial p}{\partial y} = -\frac{\partial^2 w}{\partial t^2}(x, t) \qquad (y = 0, \quad |x| < 1); \tag{7}$$

$$p \to 0 \qquad (x^2 + y^2 \to \infty); \tag{8}$$

$$\alpha \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = p(x, 0, t) \qquad (|x| < 1, \quad t > 0); \tag{9}$$

$$w(x, 0) = \gamma w_0(x), \qquad w_t(x, 0) = w_1(x) \qquad (|x| < 1, \quad t = 0); \tag{10}$$

$$w(\pm 1, t) = 0, \qquad \frac{\partial^2 w}{\partial x^2}(\pm 1, t) = 0 \qquad (|x| = 1, \quad t > 0). \tag{11}$$

The only important parameter α is equal to the ratio $M_B/(\rho L)$. In particular, $M_B = h\rho_B$ for a plate of constant thickness h with density ρ_B of the material. The parameter $\gamma = LV_p/(RV)$ characterizes the significance of the initial deflection of the plate $w(x, 0)$ for its subsequent strains and the distribution of bending stresses. For shallow waves, we have $\gamma \ll 1$. In the experiments described above, we have $h = 8$ mm, $\rho_B = 7850$ kg/m^3, and $R = 10.2$ m, which gives $\alpha = 0.2512$ and $\gamma = 0.1875$.

It follows from the determination of the parameter α and the conditions of applicability of the Euler equations of problem (1)–(3) that α is small for real materials.

The role of the weightiness of the fluid is characterized by the parameter gT^2/L, where $g = 9.81$ m/sec^2. In the case considered, this parameter is equal to 0.04, which allows one to ignore gravitational effects in the basic approximation.

Equations (5)–(8) are the hydrodynamic part of the problem, and (9)–(11) is the elastic part. The problem is bounded: the pressure on the plate $p(x,0,t)$, where $|x| < 1$, and the plate deflection $w(x,t)$ should be determined simultaneously. In dimensionless variables, the bending stresses in the plate $\sigma(x,z,t)$ are calculated by the formula

$$\sigma(x,z,t) = zw_{xx}(x,t)/2,$$

where the variable z varies over the thickness of the plate, $z = -1$ corresponds to the lower wetted part of the plate, and $z = +1$ to its upper side in the sites of the largest thickness. The upper side of the plate becomes smaller for $w_{xx}(x,t) > 0$ and is stretched for $w_{xx}(x,t) < 0$. Below, the notation $\sigma(x,t) = \sigma(x,1,t)$ is used. The scale of elastic stresses is $hEVT/L^2$. Under the experimental conditions of [1], it is equal to 841 N/mm^2.

Problem (5)–(11) is linear; therefore, it suffices to construct and investigate its solution for two cases: the functions $w_0(x)$ and $w_1(x)$ are even and odd relative to x. Here, we consider only the first case, which corresponds to a wave impact on the center of a plate.

For the solution of this problem, it seems to be natural to apply an approach within the framework of which the eigenvibration modes of a plate floating on the surface of a fluid $\Psi_m(x)$ $(m = 1, 2, \ldots)$ are first determined; the plate deflection $w(x,t)$ is then searched as a Fourier series

$$w(x,t) = \sum_{m=1}^{\infty} a_m(t)\Psi_m(x), \tag{12}$$

It is difficult to determine directly the eigenvibration modes of a floating plate (normal modes); this is possible to do from numerical calculations even in the simplest cases. The normal modes of a floating plate are of doubtless interest, because they allow one to write the solution of problem (5)–(11) as a series whose coefficients are explicitly set by the initial conditions (10).

Eigenvibration Modes of a Floating Plate. Problem (5)–(11) is solved by the method of separating the variables. According to this method, the simplest nontrivial solutions of the form

$$w(x,t) = W(t)\Psi(x), \qquad p(x,y,t) = -P(t)Q(x,y), \tag{13}$$

which satisfy all the equations of the problem, except for the initial conditions (10), are first found. The boundary condition (7) is satisfied if one sets

$$P(t) = \frac{d^2W}{dt^2}, \qquad \frac{\partial Q}{\partial y}(x,0) = \Psi(x) \quad (|x| < 1). \tag{14}$$

Substituting (13) into the equation of beam vibration (9) and taking into account (14), we obtain

$$[\alpha\Psi(x) + Q(x,0)]\frac{d^2W}{dt^2} + W(t)\frac{d^4\Psi}{dx^4} = 0.$$

The last equality leads to two relations:

$$\frac{d^2W}{dt^2} + \omega^2 W = 0; \tag{15}$$

$$\omega^2[\alpha\Psi(x) + Q(x,0)] = \frac{d^4\Psi}{dx^4} \qquad (|x| < 1), \tag{16}$$

Equation (16) should be considered together with the boundary conditions (11), which give

$$\Psi(\pm 1) = 0, \qquad \frac{d^2\Psi}{dx^2}(\pm 1) = 0, \tag{17}$$

and the boundary-value problem for the function $Q(x,y)$, which follows from relations (5), (6), (8), and (14) and has the form

$$\begin{gathered} Q_{xx} + Q_{yy} = 0 \quad (y < 0), \qquad Q = 0 \quad (y = 0, \quad |x| > 1), \\ \frac{\partial Q}{\partial y} = \Psi(x) \quad (y = 0, \quad |x| < 1), \qquad Q \to 0 \quad (x^2 + y^2 \to \infty). \end{gathered} \tag{18}$$

A linear operator that puts the function $\Psi(x)$ into correspondence with the function $Q(x,0)$ $(-1 < x < 1)$ by virtue of the boundary-value problem with the mixed boundary conditions (18) is denoted by Λ. It follows from the second Green theorem that this operator is self-conjugate and allows one to write Eq. (16) in the form

$$\omega^2[\alpha I + \Lambda]\Psi = \frac{d^4\Psi}{dx^4}, \tag{19}$$

where I is the identical operator. The operator $\alpha I + \Lambda$ is also self-conjugate; therefore, the homogeneous boundary-value problem (17), (19) has a countable set of real eigenvalues ω_j, where $\omega_{j+1} > \omega_j$ $(j = 1, 2, 3, \ldots)$ and eigenfunctions $\Psi_j(x)$, which specify the eigenvibration modes of a plate floating at the boundary of an ideal fluid. It follows from (19) and (17) that, for $i \neq j$, the functions $\Psi_j(x)$ are orthogonal in the following meaning:

$$\int_{-1}^{1} \Psi_j''(x)\Psi_i''(x)\,dx = 0. \tag{20}$$

It is convenient to introduce the new functions $\Phi_j(x)$ equal to $\Psi_j''(x)$ and orthogonal, by virtue of (20), in the usual meaning. It is clear that $\Phi_j(\pm 1) = 0$. An operator that puts the functions $\Phi_j(x)$ into correspondence with the functions $\Psi_j(x)$ is denoted by N. This operator is self-adjacent and allows one to write the solution of the problem

$$\Psi''(x) = \Phi(x) \quad (-1 < x < 1), \qquad \Psi(\pm 1) = 0 \tag{21}$$

in the form $\Psi = N\Phi$. Indeed,

$$\int_{-1}^{1} \Phi_j N\Phi_i\,dx = \int_{-1}^{1} \Psi_j''(x)\Psi_i(x)\,dx = \int_{-1}^{1} \Psi_j\Psi_i''\,dx = \int_{-1}^{1} \Phi_i N\Phi_j\,dx.$$

Using the operator N, one can rewrite Eq. (19) relative to the new function $\Phi(x) = \Psi''(x)$ in canonical form

$$A\Phi = \mu\Phi, \tag{22}$$

where $A = N(\alpha I + \Lambda)N$ and $\mu = \omega^{-2}$. The operator A is self-conjugate. To determine its eigenvalues μ_j, it is natural to pass from the operator equation (22) to an equation in the finite-dimensional space, and, thus, the problem is reduced to an eigenvalue problem of the corresponding symmetrical matrix.

The eigenfunctions of the operator A are searched for in the form

$$\Phi(x) = \sum_{i=1}^{\infty} \Phi_i\psi_i(x), \tag{23}$$

where $\psi_i(x)$ $(i = 1, 2, 3, \ldots)$ are the eigenmodes of beam vibrations in a vacuum. For a simply supported beam, we have $\psi_i(x) = \cos\lambda_i x$ and $\lambda_i = (2i-1)\pi/2$. Substituting the representation (23) into Eq. (22) and taking into account the orthogonality condition for the functions $\Phi_i(x)$, we obtain the system

$$A_1\mathbf{\Phi} = \mu\mathbf{\Phi}, \tag{24}$$

where $\mathbf{\Phi} = (\Phi_1, \Phi_2, \ldots)^t$, $A_1 = D^{-2}(\alpha I + S)D^{-2}$, $D = \text{diag}\,\{\lambda_1, \lambda_2, \ldots\}$ is the diagonal matrix, and S is the added-mass matrix [6]. The elements of the symmetrical matrix S are expressed in terms of zero- and first-order Bessel functions:

$$S_{nm} = \frac{\pi}{\lambda_n^2 - \lambda_m^2}(\lambda_n J_0(\lambda_m)J_1(\lambda_n) - \lambda_m J_0(\lambda_n)J_1(\lambda_m)) \qquad (n \neq m),$$

$$S_{nn} = \frac{\pi}{2}(J_0^2(\lambda_n) + J_1^2(\lambda_n)).$$

The elements of the matrix A_1 decrease rapidly as their numbers increase; therefore, to determine the eigenvectors and eigenvalues of this matrix, the reduction method is used. By virtue of symmetry of the matrix A_1, the eigenvectors $\mathbf{\Phi}_j$, which correspond to different eigenvalues, μ_j, are orthogonal. For clarity, we assume that the vectors $\mathbf{\Phi}_j = (\Phi_{j1}, \Phi_{j2}, \ldots)^t$ are normalized, so that

$$\sum_{i=1}^{\infty} \lambda_i^{-4} \Phi_{ji}^2 = 1, \qquad \Phi_{jj} \leqslant 0 \quad (1 \leqslant j \leqslant \infty). \tag{25}$$

The eigenmodes of plate vibrations on the fluid surface $\Psi_j(x)$ are defined as a solution of problem (21). For a simply supported plate, we have

$$\Psi_j(x) = -\sum_{i=1}^{\infty} \frac{\Phi_{ji}}{\lambda_i^2} \psi_i(x). \tag{26}$$

The functions $\Psi_j(x)$ satisfy Eq. (19) for $\omega_j = \mu_j^{-1/2}$ and the boundary conditions (17). By virtue of (25), they are normalized

$$\int_{-1}^{1} \Psi_j^2(x)\, dx = 1 \tag{27}$$

and are orthogonal in the meaning of equality (20). The relative percent contribution from the ith mode of plate vibrations in vacuum $\psi_i(x)$ to the jth vibration mode of the plate floating on the surface of an ideal and weightless fluid, $\Psi_j(x)$, is equal to $\Phi_{ji}^2 \lambda_i^{-4} \cdot 100$.

Method of Normal Modes. For the plate deflection, the solution of problem (5)–(11) is sought in the form of an expansion with respect to "wet" modes (12). Formulas (14) show that, if the coefficients $a_n(t)$ are found, the pressure profile along the plate is determined by the series

$$\begin{gathered} p(x, 0, t) = -\sum_{m=1}^{\infty} \ddot{a}_m(t) Q_m(x, 0), \\ Q_m(x, 0) = \Lambda \Psi_m(x) = \sum_{n=1}^{\infty} (\alpha \lambda_n^{-2} - \omega_m^{-2} \lambda_n^2) \Phi_{mn} \psi_n(x) \end{gathered} \tag{28}$$

by virtue of (19) and (26). Substituting (12) and (28) into (9) and taking into account (19), we obtain

$$\sum_{m=1}^{\infty} (\omega_m^{-2} \ddot{a}_m + a_m) \frac{d^4 \Psi_m}{dx^4}(x) = 0 \qquad (|x| < 1, \quad t > 0).$$

The orthogonality condition (20) gives

$$\ddot{a}_m + \omega_m^2 a_m = 0 \quad (t > 0). \tag{29}$$

The initial data (10) allow one to determine $a_m(0) = a_{m0}$ and $\dot{a}_m(0) = a_{m1}$ and to write the solution of Eq. (29) in the form

$$a_m(t) = a_{m0} \cos(\omega_m t) + \frac{a_{m1}}{\omega_m} \sin(\omega_m t). \tag{30}$$

The functions $w_0(x)$ and $w_1(x)$ in (10) are calculated in the solution of the impact problem at the initial stage [3], when the plate is wetted only partially, in the form

$$w_i(x) = \sum_{j=1}^{\infty} w_{ij} \psi_j(x), \quad i = 0,\, 1.$$

Below, the coefficients w_{ij} are considered known; as a result,

$$\begin{gathered} a_{m0} = -\gamma \sum_{j=1}^{\infty} w_{0j} \lambda_j^2 \Phi_{mj} \Big/ \sum_{j=1}^{\infty} \Phi_{mj}^2, \\ a_{m1} = -\sum_{j=1}^{\infty} w_{1j} \lambda_j^2 \Phi_{mj} \Big/ \sum_{j=1}^{\infty} \Phi_{mj}^2, \qquad m \geqslant 1. \end{gathered} \tag{31}$$

Substituting (30) and (31) into (12), we obtain

$$w(x,t) = -\sum_{j=1}^{\infty} \lambda_j^{-2} S_j(t)\psi_j(x) \tag{32}$$

for the plate deflection, and

$$\sigma(x,t) = \frac{1}{2}\sum_{j=1}^{\infty} S_j(t)\psi_j(x), \qquad S_j(t) = \sum_{m=1}^{\infty} a_m(t)\Phi_{mj} \tag{33}$$

for the distribution of bending stresses along the plate.

Complicated Support Conditions for the Plate Ends. In the boundary conditions (11), the first equality is left unchanged, while the second equality is replaced by

$$\frac{\partial^2 w}{\partial x^2}(\pm 1, t) \pm k\frac{\partial w}{\partial x}(\pm 1, t) = 0 \tag{34}$$

where k is the rigidity of a spring in dimensionless variables. The boundary condition (34) means that, as before, the plate is simply supported at the edges and, in addition, it is attached, near the edges, to the structure by spiral springs, which tend to return the plate in the equilibrium position [1]. From the applied viewpoint, condition (34) is preferred, because it allows one to model more exactly the specific features of the attachment of an elastic plate to a rigid structure by selecting k. We note that, for $k \neq 0$, the problem of the determination of the eigenvibration modes of a floating plate $\Psi_j(x)$ and the eigenfrequencies ω_j

$$\omega^2[\alpha I + \Lambda]\Psi = \frac{d^4\Psi}{dx^4} \quad (|x| < 1), \qquad \Psi(\pm 1) = 0, \qquad \Psi''(\pm 1) \pm k\Psi'(\pm 1) = 0 \tag{35}$$

is similar to problem (17), (19) and is transformed into it for $k \to 0$. The solution of problem (35) is searched in the form

$$\Psi(x) = \sum_{i=1}^{\infty} c_i\psi_i(x),$$

where $\psi_i(x)$ $(i = 1, 2, \ldots)$ are the eigenmodes of plate vibrations in a vacuum. The functions $\psi_i(x)$ depend only on the attachment conditions, and they can be written in the form

$$\psi_i(x) = A_i\left(\cos\lambda_i x - \cos\lambda_i \frac{\cosh\lambda_i x}{\cosh\lambda_i}\right) \quad (|x| < 1, \quad i = 1, 2, \ldots)$$

in the case of a central impact. Here $\lambda_i(k)$ are the solutions of the equation $2\lambda_n\cos\lambda_n + k\sin\lambda_n + k\cos\lambda_n \tanh\lambda_n = 0$, such that $0 < \lambda_1(k) < \lambda_2(k) < \ldots$. The representation $\lambda_i(k) = \lambda_i(0) + \gamma_i(k)$ $[0 \leqslant \gamma_i(k) \leqslant \pi/2]$, where the numbers $\lambda_i(0)$ correspond to the case of a simply supported beam, holds, i.e., $\lambda_i(0) = \pi(2i-1)/2$. The eigenfunctions $\psi_i(x)$ are orthogonal to each other and are normalized if one sets $A_n = [1 + \cos^2\lambda_n(\cosh^{-2}\lambda_n + 2/k)]^{-1/2}$:

$$\int_{-1}^{1} \psi_j(x)\psi_i(x)\,dx = \delta_{ij}.$$

Equality (20) does not hold for $k \neq 0$. It follows from (35) that

$$\int_{-1}^{1} \psi_j(x)\frac{d^4\psi_i}{dx^4}(x)\,dx = 0 \tag{36}$$

for $i \neq j$ and under arbitrary attachment conditions for the ends of the plate. Equality (20) is a particular case of (36), which is valid a hinged attachment and for a pinched plate.

The functions $\Psi_j(x)$ are sought for in the form $\Psi_j(x) = \sum_{i=1}^{\infty}\Psi_{ji}\psi_i(x)$; here the boundary conditions in

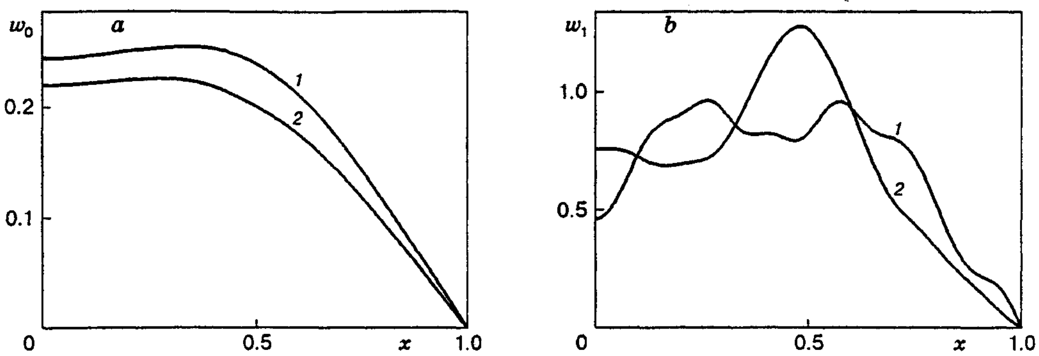

Fig. 2

problem (35) are satisfied automatically, and the equation gives

$$\alpha \Psi_{ji} + \sum_{n=1}^{\infty} S_{ni}(k)\Psi_{jn} = \omega_j^{-2}\Psi_{ji}\lambda_i^4(k), \qquad S_{ni}(k) = \int_{-1}^{1} \psi_i(x)\Lambda\psi_n(x)\,dx. \tag{37}$$

The added-mass matrix with the elements $S_{ni}(k)$ is denoted by $S(k)$, and the vector with the components $(\Psi_{j1}, \Psi_{j2}, \ldots)$ by $\mathbf{\Psi}_j$; in addition, the matrix $A_1(k) = D^{-2}(\alpha I + S(k))D^{-2}$, where $D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \}$, is introduced. Then the infinite-dimensional system (37) is rewritten in canonical form

$$A_1(k)\mathbf{\Phi}_j = \mu_j \mathbf{\Phi}_j, \tag{38}$$

where $\mathbf{\Phi}_j = -D^2\mathbf{\Psi}_j$ and $\mu_j = \omega_j^{-2}$. In Eq. (38), the matrix $A_1(k)$ is symmetrical, and its elements depend only on the parameter k. Equation (38) is transformed into Eq. (24) as $k \to 0$. The elements of the added-mass matrix $S(k)$ are calculated by the formulas

$$S_{nm}(k) = -\pi A_n A_m[\bar{S}_1(m,n,k) + \bar{S}_2(m,n,k) + \bar{S}_2(n,m,k) + \bar{S}_3(m,n,k)],$$

$$\bar{S}_1(m,n,k) = \frac{\lambda_m J_1(\lambda_m)J_0(\lambda_n) - \lambda_n J_1(\lambda_n)J_0(\lambda_m)}{\lambda_n^2 - \lambda_m^2},$$

$$\bar{S}_1(n,n,k) = -\frac{J_0^2(\lambda_n) + J_1^2(\lambda_n)}{2},$$

$$\bar{S}_2(m,n,k) = b_n\frac{\lambda_m I_1(\lambda_m)J_0(\lambda_n) + \lambda_n J_1(\lambda_n)I_0(\lambda_m)}{\lambda_n^2 + \lambda_m^2},$$

$$\bar{S}_2(n,n,k) = \frac{b_n}{2\lambda_n}(I_1(\lambda_n)J_0(\lambda_n) + J_1(\lambda_n)I_0(\lambda_n)),$$

$$\bar{S}_3(m,n,k) = b_n b_m\frac{\lambda_m I_1(\lambda_m)I_0(\lambda_n) - \lambda_n I_1(\lambda_n)I_0(\lambda_m)}{\lambda_n^2 - \lambda_m^2},$$

$$\bar{S}_3(n,n,k) = \frac{b_n^2}{2}(I_1^2(\lambda_n) - I_0^2(\lambda_n)), \qquad b_n = \frac{\cos\lambda_n \exp\lambda_n}{\cosh\lambda_n},$$

where J_0, J_1, I_0, and I_1 are the standard and transformed cylindrical functions of the zero and first orders. The nontrivial solutions of Eq. (38) are normalized in such a way that conditions (25), the form of which does not depend on the value of k, are fulfilled. The eigenvibration modes $\mathbf{\Psi}_j = D^{-2}(k)\mathbf{\Phi}_j$ satisfy equalities (27) by virtue of the adopted normalization (25).

The solution of the initial problem with complicated boundary conditions for the attachment of the ends of a plate is searched for in the form (12) and (28). For the coefficients $a_n(t)$, Eq. (29) holds, but now the orthogonality condition in the form (37) is used for its derivation. Formulas (30) and (31) remain unchanged for $k \neq 0$.

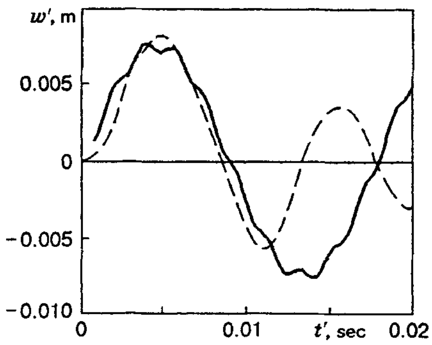

Fig. 3

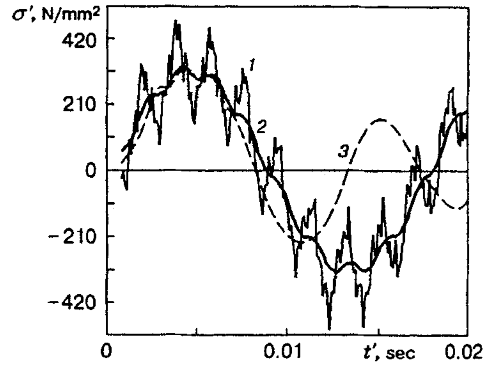

Fig. 4

Numerical Results. The vibration modes of a beam in air and on the surface of a weightless fluid were compared for the following values of the parameters: $\alpha = 0.2512$, $\gamma = 0.1875$, and $æ = 2.85$, which correspond to the experimental conditions of [1]. The fact that the corresponding modes differ little from each other was supported. The principal vibration modes coincide, and the difference does not exceed 8% between the second modes, 7% between the third modes, and 6% between the fourth modes.

The evolution of the deflection, the velocity, and the stresses at the center of the beam ($x = 0$) were studied for $æ = 2.85$. It was noted that the deflection at the center of the beam is well described by several first modes, but this is not the case for the stresses and the velocity having high-frequency fluctuations. However, the distributions of the three quantities along the beam are quite smooth.

Figure 2 shows the distributions of the deflection (a) and the velocity of the points of the beam (b) at the end of the impact stage. Curves 1 refer to the simply supported beam ends, and curves 2 to the elastic attachment ($æ = 2.85$). The elastic characteristics at the stage of immersion were calculated by direct summation of the series (32) and (33). For these parameters of the problem, it was sufficient to confine ourselves to 15 terms of the series.

The solid curve in Fig. 3 shows the evolution of the deflection at the center of the beam, and the dashed curves refer to the experimental results [1]. One can see that the calculation results agree well with the experimental data on the initial interval of time, whose duration is approximately equal to half of the basic period of vibrations of the beam lying on the fluid surface.

Figure 4 compares the calculated stresses at the center of the beam (curve 1) and the experimental data [7] (curve 3). One can see that the calculation results have the distinct peaks of high-frequency vibrations, which are absent on the experimental curve. However, the processes occurring with a frequency exceeding the limiting frequency of the gauge's sensitivity were taken into account in processing the experimental data in the integral meaning. In the calculations, this was allowed for by averaging over the limiting period. In [7], there are no sensitivity data on the gauge used, and, therefore, we took the appropriate data from [8]. In [8], to measure the stresses in a cylindrical shell at its impact on water, a gauge which records vibrations with a frequency below 5 kHz was used, which corresponds to the periods of vibrations greater than $1.25 \cdot 10^{-3}$ sec. Curve 2 was constructed by averaging the calculation results over a time interval equal to $2.1 \cdot 10^{-3}$ sec, which corresponds to the limiting frequency of the gauge's sensitivity equal to 3 kHz and results in good agreement between the calculated and experimental curves.

It is noteworthy that the thresholds of sensitivity of the pressure gauges and the gauges of relative elongation which were used in the water-impact experiments for elastic bodies differ by one order of magnitude. For example, in [7], pressure gauges with a limiting frequency of 100 kHz were applied. One can see in Fig. 4 that an increase in the sensitivity of a gauge of relative elongation can lead to a change in the experimental curve 3 when significant irregular fluctuations occur. At the stage of immersion, the pressure calculations at the center of the plate revealed its irregular time-depending fluctuations, which corresponds to the hypothesis that the calculated or measured pressures at impact cannot be used to estimate elastic characteristics [7].

The calculations of maximum stresses in the plate and at the sites where they are reached at each moment of time ($0 < t < 2$) showed that the maximum stresses are most often reached either at the center of the plate or at the points of support, but sometimes they arise on the sites where ($0.5 < |x| < 0.7$). However, the absolute maxima of stresses are reached at the center of the plate, and, therefore, Fig. 4 estimates the maximum stresses in the entire plate.

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